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International Journal of Thermal Sciences 45 (2006) 452–456
International Journal of Thermal Sciences
www.elsevier.com/locate/ijts

On the radiative transfer in a spherical medium

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Received 22 February 2005; received in revised form 7 April 2005; accepted 30 April 2005

Available online 18 January 2006

Abstract

The solution of the spherical radiative transfer problem is described for absorbing and scattering medium. The Fourier modes method is used to obtain solutions for linearly anisotropic case in a homogeneous sphere. The calculations are extended to the conservative case that set a limit for the results. It has been shown that all results converge to this limit. Based on a semi-analytic approach, numerical calculations are performed and compared to the results of several previous works.

Keywords: Radiative transfer; Spherical medium; Fourier modes solution

1. Introduction

Radiative transfer theory was developed to help astrophysicists understand light scattering from and transmitting through stellar and planetary atmospheres. Radiative transfer theory is also important in heat transfer applications. Understanding radiant heat propagation in industrial furnaces, combustion chambers or even in the design of ceramic heat exchangers is important for improving energy efficiency and for eliminating environmental contaminants. The investigation of thermal radiation heat transfer becomes essential in every high-temperature engineering applications today. Thus, the interaction of thermal radiation with participating medium must be accounted for. Many methods have been proposed and successfully applied to vast variety of radiative heat transfer problems. Solution with Chandrasekhar's X and Y functions [1], Case's normal-mode expansion technique [2–4] the spherical harmonics (P_N) [4–6] and the discrete ordinate method [5,6] can be considered as fundamental deterministic approaches to solve the radiative transfer equation.

This paper treats the problem of steady-state radiation field in a spherical medium that scatters and absorbs radiation. The spherical geometry is encountered in radiative transfer applications such as uniform core emission in stellar atmospheres, the illumination of a medium by a central isotropic point source, investigation of radiative equilibrium in a gray nonscattering medium contained between concentric black spheres, measurement of the backscattered radiances in some part of the solar spectrum.

In this study, a homogeneous sphere with linearly anisotropic scattering is considered. The radiative transfer problem with a spherical symmetry is investigated by several methods [7–19]. In this article the Fourier modes method solution is used. This paper primarily presents results for cases considered in previous works of El-Wakil et al. [19], Abulwafa [14], Siewert and Grandjean [11], and finally Siewert and Thomas [13].

In dealing with spherical medium problems, it has been shown that the spectrum of transport operator comes from antisymmetric solutions of the transport operator in a slab geometry. Therefore in the literature, the spherical geometry problems can be related to a fictitious slab problem and its odd mode solutions. This is known as pseudo-slab approach [2,11,19–21]. The pseudo-slab approach for the sphere is not required in the application of the Fourier modes method. The methodology presented here is mainly parallel to the P_N method that both yield exactly the same results. The main difference is that the eigenvalues are now found as pure imaginary in the Fourier transform space. In this study, we extend the results of previous studies to the well-known conservative case limit.

In comparison of these fundamental methods, Karp et al. [22] states that there is no consensus to consider a specific method

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doi:10.1016/j.ijthermalsci.2005.04.011

Nomenclature

A^*	albedo	
$\widehat{D}_r$	partial differential operator: $\partial/\partial r$	
$\widehat{D}_\mu$	partial differential operator: $\partial/\partial\mu$	
g_n	eigenfunction	
I	intensity of radiation	$\mathrm{W\,m^{-2}\,sr^{-1}}$
k	eigenvalue	
$K_{n+1/2}$	fractional order modified Bessel function of the third kind	
N	order of approximation	
P_n	nth order Legendre function of the first kind	
r_0	radius of sphere	m

Greek symbols

β	anisotropy factor	
δ_{ij}	Kronecker delta	
ϕ_n	expansion coefficient in P_N expansion or nth Legendre moment of the intensity	
μ	direction cosine of the propagating radiation	
ρ	density of radiation intensity	$\mathrm{W\,m^{-2}\,sr^{-1}}$
ω	single scattering albedo	

Subscript, superscript

0	fixed radial variable for radius
n	polynomial degree
r	radial component

superior to others. With a reasonable justification for this diversity, certain methods are better in certain domains than others. However, the method presented here provides a direct approach for a spherical medium without requiring a theory which establishes a relation between the spectrum of transport operator in plane and spherical geometry (pseudo-slab approach). Furthermore, the conservative case is treated in this scheme effectively.

2. Radiative transfer for non-conservative case

The radiative transfer equation for an absorbing and linearly anisotropic scattering sphere is given by

$$\mu\frac{\partial I(r,\mu)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial\mu}+I(r,\mu)=\frac{\omega}{2}\int_{-1}^{1}\mathrm{d}\mu'(1+\beta\mu\mu')I(r,\mu'),\quad r\in(0,r_0)\tag{1}$$

where I is the intensity of radiation, μ direction cosine of the propagating radiation ω single scattering albedo β anisotropy factor and the rest of the parameters is given in the standard notation. The spherical medium is subject to the boundary condition

$$I(r_0,-\mu)=1,\quad \mu\geqslant 0\tag{2}$$

Eq. (1) is not amenable to the separation of variables. Thus, taking the Fourier transform directly is not possible here in contrast to the plane geometry case. In Eq. (1), the partial differential operators: $\widehat{D}_r=\partial/\partial r$ and $\widehat{D}_\mu=\partial/\partial\mu$ commute, that is, $\widehat{D}_r\widehat{D}_\mu I(r,\mu)=\widehat{D}_\mu\widehat{D}_r I(r,\mu)$. In this case, we can treat $\widehat{D}_r$ as constant in the solution procedure (obviously, this does not mean that our final solution will be r independent; see Ref. [23, p. 261]). Thus we rewrite Eq. (1) in the form

$$\left(1-\mu^2\right)\frac{\partial I(r,\mu)}{\partial\mu}+r[\mu\widehat{D}_r+1]I(r,\mu)=\frac{\omega r}{2}\int_{-1}^{1}\mathrm{d}\mu'(1+\beta\mu\mu')I(r,\mu')\tag{3}$$

We here expand the specific intensity into a series of Legendre polynomials $P_n(\mu)$ which form a complete orthogonal set within the range $(-1,1)$. We then consider a solution for this integro-differential equation in the form

$$I(r,\mu)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\phi_n(r)P_n(\mu)\tag{4}$$

We then substitute Eq. (4) into Eq. (3), multiply by $P_m(\mu)$ and integrate over μ and use the recurrence relations

$$\mu P_m(\mu)=\left[\frac{(m+1)}{2m+1}P_{m+1}(\mu)+\frac{m}{2m+1}P_{m-1}(\mu)\right]\tag{5a}$$

$$\left(\mu^2-1\right)\frac{\partial P_m(\mu)}{\partial\mu}=\frac{m(m+1)}{2m+1}\left[P_{m+1}(\mu)-P_{m-1}(\mu)\right]\tag{5b}$$

and orthogonality relations of the Legendre functions

$$\int_{-1}^{1}\mathrm{d}\mu P_n(\mu)P_m(\mu')=\frac{2}{2n+1}\delta_{nm}\tag{6}$$

to obtain

$$(n+1)\left[\widehat{D}_r+\frac{n+2}{r}\right]\phi_{n+1}(r)+n\left[\widehat{D}_r-\frac{n-1}{r}\right]\phi_{n-1}(r)+(2n+1)\left(1-\omega\delta_{0n}-\frac{\omega}{9}\beta\delta_{1n}\right)\phi_n(r)=0\tag{7}$$

This equation is now in an appropriate form for Fourier–Bessel transform. Thus we assume

$$\phi_n(r)=g_n(k)f_n(k,r)\tag{8}$$

where

$$f_n(k,r)=\frac{K_{n+1/2}(-ikr)}{\sqrt{-ikr}}\tag{9}$$

$f_n(k,r)$ satisfies the following recurrence relations

$$ikf_n(k,r)=\frac{\partial f_{n+1}(k,r)}{\partial r}+\frac{n+2}{r}f_{n+1}(k,r)\tag{10a}$$

$$ikf_n(k,r)=\frac{\partial f_{n-1}(k,r)}{\partial r}-\frac{n-1}{r}f_{n-1}(k,r)\tag{10b}$$

Using these results, it can be seen that eigenfunctions, $g_n(k)$ satisfy the recurrence relation in the form

$$ik(n+1)g_{n+1}(k) + (2n+1)g_n(k) + ikng_{n-1}(k) - \omega g_0(k)\delta_{n0} - \frac{\omega\beta}{3} g_1(k)\delta_{n1} = 0 \quad (11)$$

Hence the required solution becomes

$$I(r,\mu) = \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) \sum_{j=1}^{(N+1)/2} g_n(k_j) \times \left[A_j \frac{K_{n+1/2}(-ikr)}{\sqrt{-ikr}} + (-1)^n B_j \frac{K_{n+1/2}(ikr)}{\sqrt{ikr}} \right] \quad (12)$$

where we have used $g_n(-k_j) = (-1)^n g_n(k_j)$. Then Eq. (12) is used in the symmetry condition, $I(r,\mu) = I(-r,-\mu)$ to obtain $B_j = A_j$. Employing the boundary condition given by Eq. (2) yields

$$\sum_{n=0}^{N} \frac{2n+1}{2} (-1)^n P_n(\mu) \sum_{j=1}^{(N+1)/2} g_n(k_j) \times \left[\frac{K_{n+1/2}(-ikr_0)}{\sqrt{-ikr_0}} + (-1)^n \frac{K_{n+1/2}(ikr_0)}{\sqrt{ikr_0}} \right] A_j = 1 \quad (13)$$

Multiplying both sides by $P_{2i-1}(\mu)$ i.e., $i = 1, \ldots, (N+1)/2$ and integrating over $\mu(0,1)$ give a linear system to determine expansion coefficients, $\{A_j\}$. With a definition

$$\theta_{ij} = \int_0^1 d\mu P_{2i-1}(\mu) P_j(\mu) \quad (14)$$

this linear system is given by

$$\sum_{n=0}^{N} \frac{2n+1}{2} (-1)^n \sum_{j=1}^{(N+1)/2} \theta_{ij} g_n(k_j) \times \left[\frac{K_{n+1/2}(-ikr_0)}{\sqrt{-ikr_0}} + (-1)^n \frac{K_{n+1/2}(ikr_0)}{\sqrt{ikr_0}} \right] A_j = \theta_{i0}, \quad i = 1, \ldots, (N+1)/2 \quad (15)$$

Following the determination of unknown expansion coefficients, we can compute the albedo

$$A^* = 2\int_0^1 d\mu\, \mu I(r_0,\mu) \quad (16)$$

and the density

$$\rho(r) = \int_{-1}^1 d\mu I(r,\mu) \quad (17)$$

3. Radiative transfer for conservative case

In the radiative equilibrium case, we have $\omega = 1$ and the solution is constant throughout the sphere, namely $I(r,\mu) = 1$. This solution satisfies Eq. (1). Following the standard procedure, we can also verify this easily. Here a pair of eigenvalues is zero. Thus we have to modify the procedure of non-conservative case. Then instead of Eq. (8), we assume a solution in the form

$$\phi_n(r) = \left(C_0 + 3\frac{C_1}{r} \right)\delta_{n0} + \frac{C_1}{r^2}\delta_{n1} + \frac{g_n(k)K_{n+1/2}(-ikr)}{\sqrt{-ikr}} \quad (18)$$

If Eq. (18) has an appropriate form, when this is substituted into Eq. (7), the optical variable should disappear. Indeed this is the case, we thus obtain again Eq. (11) with $\omega = 1$. Then the solution becomes

$$I(r,\mu) = \frac{C_0}{2} + \frac{3}{2}\frac{C_1}{r}\left(1 + \frac{\mu}{r}\right) + \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) \sum_{j=1}^{(N+1)/2} g_n(k_j) \times \left[A_j \frac{K_{n+1/2}(-ikr)}{\sqrt{-ikr}} + (-1)^n B_j \frac{K_{n+1/2}(ikr)}{\sqrt{ikr}} \right] \quad (19)$$

Using the symmetry property, we find that $C_1 = 0$ and $B_j = A_j$. Then proceeding exactly as in the non-conservative case, we eventually obtain the linear system

$$\frac{C_0}{2}\theta_{i0} + \sum_{n=0}^{N} \frac{2n+1}{2} (-1)^n \sum_{j=1}^{(N+1)/2} \theta_{ij} g_n(k_j) \times \left[\frac{K_{n+1/2}(-ikr_0)}{\sqrt{-ikr_0}} + (-1)^n \frac{K_{n+1/2}(ikr_0)}{\sqrt{ikr_0}} \right] A_j = \theta_{i0} \quad (20)$$

Solving this system, we find that $C_0 = 2$ and $A_j = 0$ for $j = 1, \ldots, (N+1)/2$. The albedo and density are evaluated again by using Eqs. (16) and (17) and thus, limit values $A^* = 1$ and $\rho(r) \equiv \text{const} = 2$ are found.

4. Numerical results

A computer program was written for the Fourier mode solution to compute the albedo and density for linearly anisotropic scattering in a homogeneous spherical medium. For this purpose, we developed a code using MATHEMATICA software. To be able to implement the procedure, we first need to know eigenvalues and eigenvectors. The eigenvalues are determined from roots of $\det|\mathbf{M}(k)|$. The bracketing is performed to locate the interval $[k_1, k_2]$ which contains the root of the equations. Here $k_1 < k_j < k_2$ and if $f(k) = \det|\mathbf{M}(k)|$, then sign $f(k_1) \neq$ sign $f(k_2)$. Then the regular-false method is used to determine an eigenvalue of the Fourier modes [24] as follows

$$k_{ji+1} = \frac{k_{2i} f(k_{1i}) - k_{1i} f(k_{2i})}{k_{2i} - k_{1i}} \quad (21)$$

where i is the iteration step for the virtual eigenvalue. Checking sign k_{ji+1} is replaced either by k_{1j} or k_{2j} and iteration continues until the result converges to the required virtual eigenvalue.

Table 1
The albedo A^* of a spherical medium for various r_0 and ω

ω	r_0	Present	Ref. [19][a]	Refs. [11,14][b]
0.10	0.1	0.88774	0.87576	0.88766
0.20		0.88949	0.89907	0.89940
0.30		0.91120	0.91099	0.91129
0.40		0.92350	0.92308	0.92340
0.50		0.93443	0.93534	0.93564
0.60		0.94823	0.94778	0.94816
0.70		0.96096	0.96038	0.96079
0.80		0.97376	0.97313	0.97363
0.90		0.98677	0.98587	0.98640
0.95		0.99333	0.99092	0.99333
0.99		0.99868		
0.999		0.99987		
1.0		1.0		
0.10	0.5	0.56295	0.56295	0.5628
0.20		0.59948	0.59950	0.5995
0.30		0.63857	0.63838	0.6384
0.40		0.67990	0.67981	0.6798
0.50		0.72446	0.72405	0.7240
0.60		0.77133	0.77141	0.7714
0.70		0.82224	0.82224	0.8222
0.80		0.87684	0.87695	0.8769
0.90		0.93605	0.93602	0.9360
0.95		0.96735	0.96737	0.9673
0.99		0.99374		
0.999		0.99934		
1.0		1.0		
0.10	1.0	0.33681	0.33680	0.33685
0.20		0.38063	0.38024	
0.30		0.42874	0.42831	0.42884
0.40		0.48203	0.48179	
0.50		0.54251	0.54168	0.54226
0.60		0.60963	0.60927	
0.70		0.68669	0.68619	0.68657
0.80		0.77473	0.77469	
0.90		0.87849	0.87779	0.87801
0.95		0.93630	0.93608	
0.99		0.98685		
0.999		0.99867		
1.0		1.0		
0.10	2.0	0.14888	0.14804	0.14904
0.20		0.18892	0.18728	0.18906
0.30		0.23462	0.23243	0.23462
0.40		0.28739	0.28494	0.28722
0.50		0.34877	0.34680	0.34890
0.60		0.42256	0.42098	0.42270
0.70		0.51369	0.51208	0.51330
0.80		0.62833	0.62771	0.62839
0.90		0.78164	0.78140	0.78161
0.95		0.88002	0.88001	0.88004
0.99		0.97396		
0.999		0.99734		
1.0		1.0		
0.10	3.0	0.08574	0.08476	0.08590
0.20		0.12061	0.11914	
0.30		0.16144	0.15919	0.16206
0.40		0.20938	0.20648	
0.50		0.26570	0.26315	0.26595
0.60		0.33487	0.33263	
0.70		0.42229	0.42077	0.42244
0.80		0.53955	0.53862	
0.90		0.71009	0.71000	0.71029
0.95		0.83233	0.83243	
0.99		0.96151		
0.999		0.99602		
1.0		1.0		
0.10	5.0	0.04716	0.04569	0.0473
0.20		0.07830	0.07547	0.0782
0.30		0.11393	0.11042	0.1139
0.40		0.15553	0.15187	0.1554
0.50		0.20554	0.20178	0.2051
0.60		0.26631	0.26342	0.2662
0.70		0.34493	0.34267	0.3448
0.80		0.45324	0.45178	0.4531
0.90		0.62350	0.62292	0.6234
0.95		0.76249	0.76236	0.7625
0.99		0.93854		
0.999		0.99338		
1.0		1.0		

[a] Based on the reported results of the highest approximation, $N = 6$.
[b] $r_0 = 1.0$ and 3.0 results are based on the Ref. [11].

Also this procedure is repeated until all eigenvalues are obtained. They are found in pairs $\pm k_j$ for $j = 1, \ldots, (N+1)/2$. The eigenvectors or actually numerical values of $g_n(k_j)$ are evaluated from Eq. (11) in the form

$$g_n(k_j) = \frac{(2n-1)i}{k_j} g_{n-1}(k_j) - (n-1)g_{n-2}(k_j) - i\frac{\omega g_0(k_j)\delta_{n0}}{k_j} \tag{22}$$

The unknown expansion coefficients, $\{A_j\}$, $j = 1, \ldots, (N+1/2)$, are determined by solving linear systems of Eqs. (15) and (20). The coefficient matrices in these linear systems are always full matrices. The Gauss-elimination method is used as solution method [25]. Finally, the albedo and density are evaluated from the Eqs. (16) and (17), respectively.

The only isotropic scattering case is considered here to compare the results those of the literature. The calculated albedo values are tabulated in Table 1. In this table various order approximation results are compared with those of Refs. [11,14,19]. The numerical results are given for different sphere radii, and single scattering albedo. The single scattering albedo values are increased near to the conservative case and finally the conservative case results are given.

In Figs. 1 and 2 we plotted the variation of the density $\rho(r)$ with r for sphere radii $r_0 = 1.0$ and $r_0 = 10.0$, respectively. These results are in agreement with the data tabulated in Ref. [13]. We can also see from these figures how this variation converges to the conservative case data when r_0 is increased. The comparison of the results shows that the Fourier mode method accurately evaluates the albedo and the density.

5. Concluding remark

In this article, the radiative transfer problem is treated for a homogeneous sphere. The medium absorbs and scatters the radiation. Fourier modes method is considered as a solution

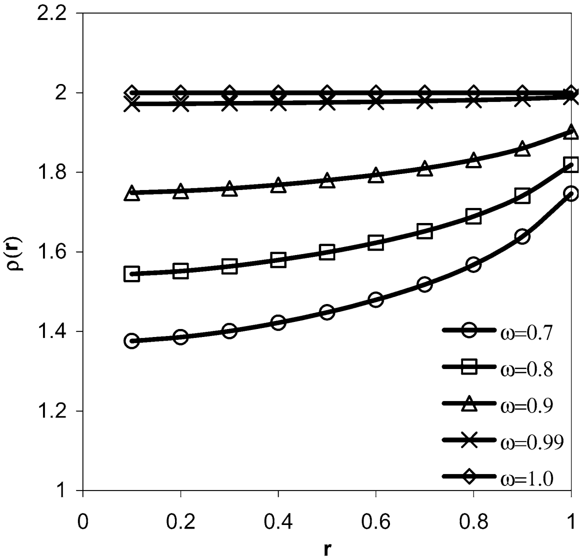

Fig. 1. The density $\rho(r)$ for a sphere with radius $r_0 = 1.0$.

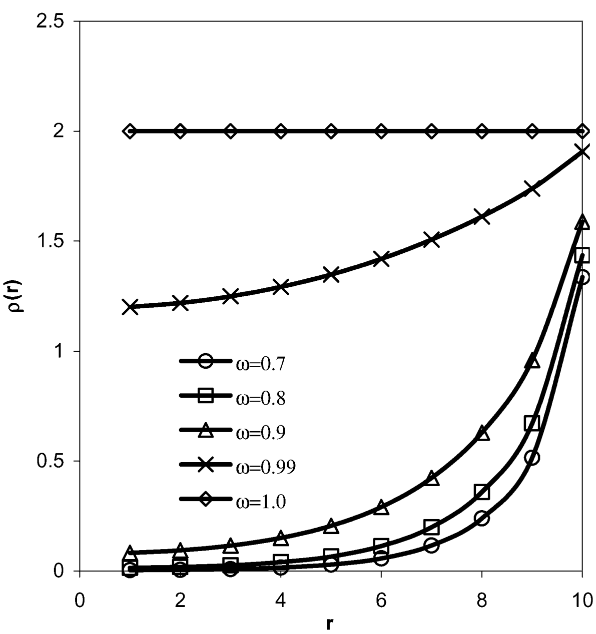

Fig. 2. The density $\rho(r)$ for a sphere with radius $r_0 = 10.0$.

method. It is important to note that this method results are identical to those of the spherical harmonics method (P_N). This includes the equivalence of the same order N results. Thus main difference arises in the eigenvalue domain that while the P_N method eigenvalues are real, they are now found as pure imaginary. In the Fourier modes method, a special step is required to cast the integro-differential equation into an appropriate form for the Fourier–Bessel transform. Therefore, the partial differential operator is treated as constant. Finally both methods do not require using the pseudo-slab approach. Although, the analytical solution for the conservative case in a spherical medium is trivial, the same methodology can be applied also to find this case solution.

Acknowledgement

The author is grateful to Dr. M. Aydin, Evora University, Portugal for many useful comments and suggestions.

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